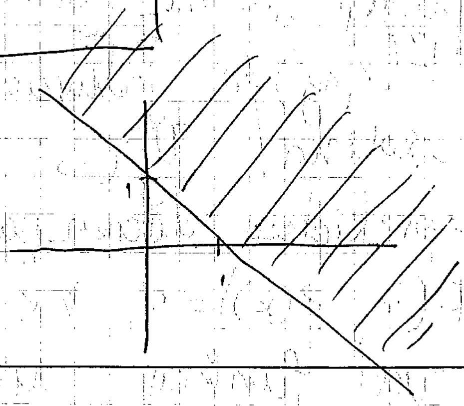
Oblast definisanosti fja VP

(1.) $f(x,y) = \sqrt{1+x+y}$

$\mathcal{D} = \{(x,y) \mid 1+x+y \geq 0\}$

$y \geq -x-1$

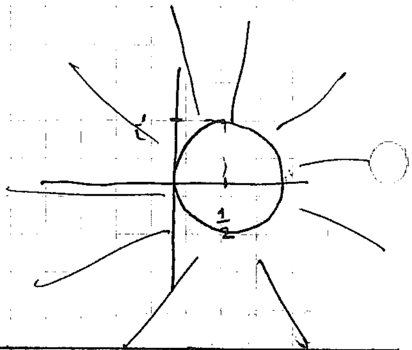


(2) $f(x,y) = \sqrt[4]{x^2-x+y^2}$

$\mathcal{D} = \{(x,y) \mid x^2-x+y^2 \geq 0\}$

$x^2 - \frac{1}{2}\cdot 2x + \frac{1}{4} - \frac{1}{4} + y^2 \geq 0$

$\left(x-\frac{1}{2}\right)^2 + y^2 \geq \left(\frac{1}{2}\right)^2$



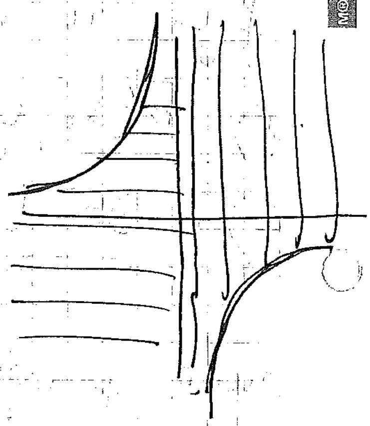
(3.) ~~$\mathcal{D} = \{ x$~~

$f(x,y) = \ln(1+xy)$

$\mathcal{D} = \{(x,y) \mid 1+xy > 0\}$

$xy > -1$ 3 slučaja !

$x>0$
$x=0$
$y<0$



1° $x > 0 \rightarrow y > -\frac{1}{x}$

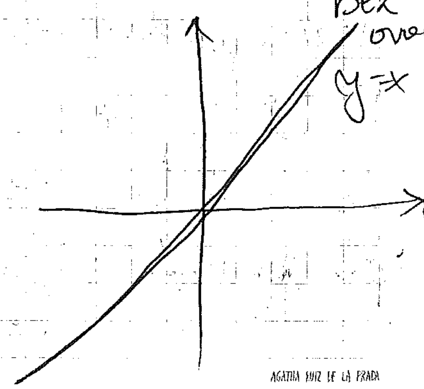
2° $x = 0 \rightarrow 1+0\cdot y = 1 > 0$

3° $x < 0 \rightarrow y < -\frac{1}{x}$

(4.) $f(x,y) = \dfrac{x^2+3y^2}{x-y} \rightarrow x-y \neq 0$

$\mathcal{D} = \{(x,y) \in \mathbb{R}^2 \mid x \neq y\}$

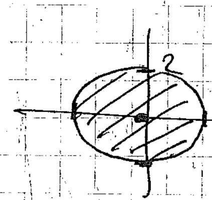
bez ovog $y=x$



5. $f(x,y) = \sqrt{4 - x^2 - y^2}$

$4 - x^2 - y^2 \geq 0 \longrightarrow x^2 + y^2 \leq 4$

$D = \{ (x,y) \mid x^2 + y^2 \leq 4 \}$



6. $f(x,y) = \sqrt{1 - x^2} + \sqrt{y^2 - 1}$

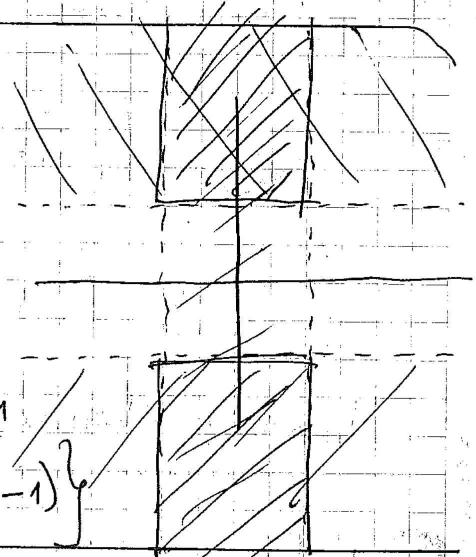
$1 - x^2 \geq 0 \quad \wedge \quad y^2 - 1 \geq 0$

$x^2 \leq 1 \quad \wedge \quad y^2 \geq 1$

$|x| < 1 \quad \wedge \quad |y| \geq 1$

$-1 \leq x \leq 1 \quad \wedge \quad 1 \leq y \leq -1$

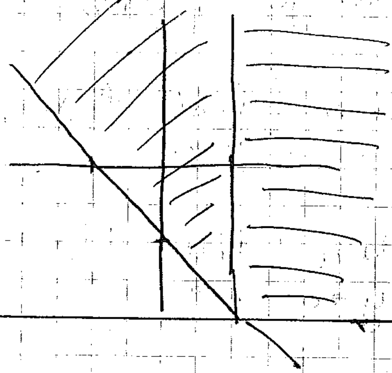
$D = \{ (x,y) \mid -1 \leq x \leq 1 \wedge (y \geq 1 , y \leq -1) \}$



7. $f(x,y) = \dfrac{\sqrt{x + y + 1}}{x - 1}$

$x - 1 \neq 0 \quad \wedge \quad x + y + 1 \geq 0$

$x \neq +1 \quad \wedge \quad y \geq -x - 1$

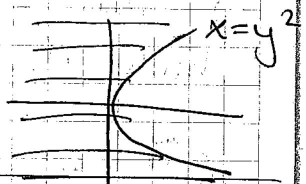


8. $f(x,y) = x \cdot \ln(y^2 - x)$

$y^2 - x > 0$

$x < y^2$

$D = \{ (x,y) \mid x < y^2 \}$



$\boxed{\text{Granične vrijednosti. Neprekidnost}}$

9. Izračunati granične vrijednosti

a) $\lim\limits_{n \to \infty} \left(\frac{1}{n}, -\frac{1}{n} \right)$

$X_n = \left(\frac{1}{n}, -\frac{1}{n} \right) \qquad X_2 = \left(\frac{1}{2}, -\frac{1}{2} \right)$

$X_1 = (1, -1) \qquad X_3 = \left(\frac{1}{3}, -\frac{1}{3} \right)$

$$\lim_{n\to\infty}\left(\frac{1}{n}, -\frac{1}{n}\right) = \left(\lim_{n\to\infty}\frac{1}{n}, -\lim_{n\to\infty}\frac{1}{n}\right) = (0,0)$$

$\to$ ovaj niz teži ka tački $(0,0)$

b) $\lim_{n\to\infty}\left(\left(\frac{n}{n+1}\right)^n, \frac{1}{n}-1\right) =$

$$= \left(\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n, \lim_{n\to\infty}\left(\frac{1}{n}-1\right)\right) =$$

$$= \left(\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^n\right)^{-1}, \lim_{n\to\infty}\left(\frac{1}{n}-1\right)\right) = \left(\frac{1}{e}, -1\right)$$

$\hookrightarrow$ niz konvergira ka tački $\left(\frac{1}{e}, -1\right)$

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10. Neka je $f(x,y) = \dfrac{x^2 \cdot y^2}{x^2y^2+(x-y)^2}$. Dokazati da:

a) $\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = 0$;

b) $\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = 0$;

c) da fja nema gr. vr. u tački $(0,0)$ iako su naizmjenični limesi jednaki

a) $$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{x^2\cdot y^2}{x^2\cdot y^2+(x-y)^2}\right) = \lim_{x\to 0}\left(\frac{0}{\underbrace{0+x^2}_{0}}\right) = \lim_{x\to 0} 0 = 0$$

b) $$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^2\cdot y^2}{x^2y^2+(x-y)^2}\right) = 0$$

$$\underbrace{\exists \lim_{x\to x_0} f(x) = A \Rightarrow \left((x_n)\to x_0 \Rightarrow f(x_n)\to A\right), \text{ za } \forall (x_n)\to x_0}_{\text{Hajneovi nizovi}}$$

$\to$ limes postoji ukoliko za svaki niz koji teži ka x_0 važi da je $f(x_n) = A$

→ Hajneove nizove koristimo kada želimo da dokažemo da limes ne postoji → nađemo 2 niza koja teže ka x_0, a $f(x_n) \neq f(y_n)$

$\nexists \lim_{x \to x_0} f(x_0)$

→ po Hajneovoj definiciji ovo mora da je jednako da bi to bila gr. vr.

$(x_n) \to x_0 \quad \lim_{n\to\infty} f(x_n) = A_1$

$(y_n) \to x_0 \quad \lim_{n\to\infty} f(y_n) = A_2$

$A_1 \neq A_2$

c) uzmimo $x_n = \left(\frac{1}{n}, \frac{1}{n}\right)$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \left(\frac{1}{n}, \frac{1}{n}\right) = (0,0)$$

neka je $y_n = \left(\frac{1}{n}, -\frac{1}{n}\right)$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \left(\frac{1}{n}, -\frac{1}{n}\right) = (0,0)$$

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f\left(\frac{1}{n}, \frac{1}{n}\right) = \lim_{n\to\infty} \left(\frac{\frac{1}{n^2} \cdot \frac{1}{n^2}}{\frac{1}{n^2} \cdot \frac{1}{n^2} + \left(\frac{1}{n} - \frac{1}{n}\right)^2} \right) = 1$$

$$\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} f\left(\frac{1}{n}, -\frac{1}{n}\right) = \lim_{n\to\infty} \frac{+\frac{1}{n^2} \cdot \frac{1}{n^2}}{\frac{1}{n^2} \cdot \left(-\frac{1}{n}\right)^2 + \left(\frac{1}{n} + \frac{1}{n}\right)^2} = 0$$

$$\lim_{n\to\infty} \frac{1}{1+4n^2}$$

* Imamo 2 niza (x_n) i (y_n) u $\mathbb{R}^2$ i oba teže ka tački $(0,0)$ ali $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$ pa zaključujemo da ne postoji $\lim_{(x,y)\to(0,0)} f(x,y)$.

Da je bilo ispitati neprekidnost - dokazati da fja nije neprekidna $\underbrace{f(x,y)}_{\text{u tački } (0,0)} = \frac{x^2y^2}{x^2y^2 + (x-y)^2}$

→ $\lim_{x\to x_0} f(x) = f(x_0)$

↳ ako pokažemo da gr. vr. fje u tački ne postoji → ne možemo govoriti o neprekidnosti → nije neprekidna

11. Izračunati granične vrijednosti:

a) $\lim\limits_{(x,y)\to(0,a)} \frac{\sin xy}{x}$; b) $\lim\limits_{\substack{x\to+\infty \\ y\to+\infty}} \frac{x^2+y^2}{x^4+y^4}$; c) $\lim\limits_{(x,y)\to(+\infty,+\infty)} \frac{x^2-y^2}{x^4+y^4}$

a) $\lim\limits_{(x,y)\to(0,a)} \frac{\sin xy}{x} = \lim\limits_{(x,y)\to(0,a)} \frac{\sin xy}{xy} \cdot y = \left[\begin{array}{l} \text{ideja smjene} \\ \text{neka je } xy = t \\ (x,y)\to(0,a) \\ t\to 0 \end{array} \right] =$

$= \lim\limits_{t\to 0} \frac{\sin t}{t} \cdot \lim\limits_{\substack{x\to 0 \\ y\to a}} y = 1\cdot a = a$

b) teorema o ukliještenju

$0 \le \frac{x^2+y^2}{x^4+y^4} = \frac{x^2}{x^4+y^4} + \frac{y^2}{x^4+y^4} \le \frac{x^2}{x^4} + \frac{y^2}{y^4} = \frac{1}{x^2} + \frac{1}{y^2}$ (1)

$\lim\limits_{\substack{x\to+\infty \\ y\to+\infty}} \left(\frac{1}{x^2} + \frac{1}{y^2}\right) = 0$ (2)

iz (1) i (2) na osnovu teoreme o ukliještenju slijedi da je $\lim\limits_{\substack{x\to+\infty \\ y\to+\infty}} \frac{x^2+y^2}{x^4+y^4} = 0$

c) Napomena:

1° Ako je $\lim\limits_{(x,y)\to(x_0,y_0)} |f(x,y)| = 0$, tada je i $\lim\limits_{\substack{x\to x_0 \\ y\to y_0}} f(x,y) = 0$

2° $|a+b| \le |a| + |b|$

$|a-b| \le |a| + |b|$

$0 \le \left|\frac{x^2-y^2}{x^4+y^4}\right| = \left|\frac{x^2}{x^4+y^4} + \frac{-y^2}{x^4+y^4}\right| \le \left|\frac{x^2}{x^4+y^4}\right| + \left|\frac{y^2}{x^4+y^4}\right| =$

→ sve pozitivno → možemo da se oslobodimo aps. vr.

$$= \frac{x^2}{x^4+y^4} + \frac{y^2}{x^4+y^4} \le \frac{x^2}{x^4} + \frac{y^2}{y^4} = \frac{1}{x^2} + \frac{1}{y^2} \quad (1)$$

$$\lim_{\substack{x \to +\infty \\ y \to +\infty}} \left(\frac{1}{x^2} + \frac{1}{y^2} \right) = 0 \quad (2)$$

Iz (1) i (2) $\Rightarrow \lim\limits_{\substack{x \to +\infty \\ y \to +\infty}} |f(x,y)| = 0$ pa je i $\lim\limits_{\substack{x \to +\infty \\ y \to +\infty}} f(x,y) = 0$

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12 Naći $\lim\limits_{(x,y)\to(0,2)} (1+xy)^{\frac{2}{x^2+xy}}$

$$\lim_{(x,y)\to(0,2)} (1+xy)^{\frac{2}{x^2+xy}} = \lim_{\substack{x\to 0 \\ y \to 2}} \left((1+xy)^{\frac{1}{xy}} \right)^{\frac{2xy}{x^2+xy}} =$$

$$= \lim_{\substack{x\to 0 \\ y \to 2}} \left((1+xy)^{\frac{1}{xy}} \right)^{\frac{2y}{x+y}} \overset{(*)}{=} e^{\lim\limits_{\substack{x\to 0 \\ y \to 2}} \frac{2y}{x+y}} = e^2$$

(*) $\lim\limits_{\substack{x\to 0 \\ y \to 2}} (1+xy)^{\frac{1}{xy}} = \left[\begin{array}{l} xy = t \\ (x,y) \to (0,2) \\ \Rightarrow t \to 0 \end{array} \right] = \lim\limits_{t \to 0} (1+t)^{\frac{1}{t}} = e$

---

13 Naći $\lim\limits_{\substack{x \to +\infty \\ y \to +\infty}} \underbrace{(x^2+y^2)}_{\ge 0} \cdot \underbrace{e^{-(x+y)}}_{\ge 0}$ Ograničavamo

$$0 \le (x^2+y^2) \cdot e^{-(x+y)} \le (x^2+2xy+y^2) \cdot e^{-(x+y)} =$$

$$= (x^{\cancel{2}}+y)^2 \cdot e^{-(x+y)} \quad (1) \qquad \frac{\text{polinom}}{\text{exp}} \to 0$$

$$\lim_{\substack{x \to +\infty \\ y \to +\infty}} (x+y)^2 \cdot e^{-(x+y)} = \left[\begin{array}{l} x+y = t \\ (x,y) \to (+\infty, +\infty) \\ \Rightarrow t \to +\infty \end{array} \right] =$$

$$= \lim_{t \to +\infty} t^2 \cdot e^{-t} = \lim_{t \to +\infty} \frac{t^2}{e^t} = 0 \quad (2)$$

Iz (1) i (2) $\Rightarrow \lim\limits_{\substack{x \to +\infty \\ y \to +\infty}} f(x,y) = 0$

na osnovu teoreme o uklještenju

MIQUELRIUS
AGATHA RUIZ DE LA PRADA

14. Naći $\lim\limits_{\substack{x \to +\infty \\ y \to +\infty}} \left(\frac{xy}{x^2+y^2}\right)^{x^2}$

Ograničavamo:

$(x-y)^2 \geq 0$

$x^2 - 2xy + y^2 \geq 0 \quad \to \quad x^2+y^2 \geq 2xy \quad / \cdot \frac{1}{x^2+y^2}$

$1 \geq \frac{2xy}{x^2+y^2} \quad / \cdot \frac{1}{2}$

$\frac{1}{2} \geq \frac{xy}{x^2+y^2} \quad \to \quad \frac{xy}{x^2+y^2} \leq \frac{1}{2}$

$0 \leq \left(\frac{xy}{x^2+y^2}\right)^{x^2} \leq \left(\frac{1}{2}\right)^{x^2} \quad (1)$

$\lim\limits_{(x,y) \to (+\infty,+\infty)} \left(\frac{1}{2}\right)^{x^2} = \left[\lim\limits_{n \to \infty} q^n = 0 \text{ ako } |q| < 1\right] = 0 \quad (2)$

Iz (1) i (2) na osnovu teoreme o uklještenju slijedi $\Rightarrow \lim\limits_{\substack{x \to +\infty \\ y \to +\infty}} \left(\frac{xy}{x^2+y^2}\right)^{x^2} = 0$

---

15. Ispitati da li postoje:

a) $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \frac{x^2-y^2}{x^2+y^2}$ $\to$ cilj pokazati da f-ja nema gr. vr. u $(0,0)$

$f(x,y) = \frac{x^2-y^2}{x^2+y^2}$

$\to$ Neka je $x_n = \left(\frac{1}{n}, \frac{1}{n}\right)$ i $y_n = \left(\frac{1}{n}, \frac{2}{n}\right)$

$\lim\limits_{n \to \infty} x_n = \lim\limits_{n \to \infty} \left(\frac{1}{n}, \frac{1}{n}\right) = (0,0)$

$$\lim_{n\to\infty}(y_n) = \lim_{n\to\infty}\left(\frac{1}{n}, \frac{2}{n}\right) = (0,0)$$

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f\left(\frac{1}{n}, \frac{1}{n}\right) = \lim_{n\to\infty} \frac{\frac{1}{n^2} - \frac{1}{n^2}}{\frac{1}{n^2} + \frac{1}{n^2}} = 0$$

Konst. niz.

$$\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} f\left(\frac{1}{n}, \frac{2}{n}\right) = \lim_{n\to\infty} \frac{\cancel{\frac{1}{n^2}} - \frac{4}{n^2}}{\frac{5}{n^2}} = -\frac{3}{5}$$

Konstantan niz

1° $\lim_{n\to\infty} x_n = (0,0) = \lim_{n\to\infty}(y_n)$

2° $\underbrace{\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)}$

ne postoji gr. vr. $\lim_{(x,y)\to(0,0)} f(x,y)$ → fja nije neprekidna

b) $\lim_{x\to 0}\left(\lim_{y\to 0} \frac{x^2-y^2}{x^2+y^2}\right) = 1$

$\lim_{y\to 0}\left(\lim_{x\to 0} \frac{x^2-y^2}{x^2+y^2}\right) = -1$

$\Rightarrow$ nema gr. vr. u $(0,0)$

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16. Naći $\lim_{(x,y)\to(+\infty,+\infty)} \frac{x+y}{x^2+xy+y^2}$

$$0 \le \frac{x+y}{x^2+xy+y^2} \le \frac{x+y}{x\cdot y} = \frac{x}{xy} + \frac{y}{xy} = \frac{1}{y} + \frac{1}{x} \quad (1)$$

$$\lim_{\substack{x\to\infty\\ y\to\infty}} \left(\frac{1}{y} + \frac{1}{x}\right) = 0 \quad (2)$$

iz (1) i (2) slijedi da je $\lim_{\substack{x\to+\infty\\ y\to+\infty}} \frac{x+y}{x^2+xy+y^2} = 0$

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(17) Odrediti graničnu vrijednost niza:

a) $X_n = \left(\frac{2n+3}{5n+4}, \frac{n^2+n}{n^3-n} \right)$

b) $X_n = \left(\left(\frac{n}{n+1}\right)^n, \frac{1}{n} - 1, \frac{3^n+7^n}{-3^n+7^n+1} \right)$

c) $X_n = \left(\frac{1}{n}, \frac{1}{n^2} \right)$

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c) $\lim_{n\to\infty} x_n = \left(\lim_{n\to\infty} \frac{1}{n}, \lim_{n\to\infty} \frac{1}{n^2} \right) = 0$

a) $\lim_{n\to\infty} x_n = \left(\lim_{n\to\infty} \frac{2n+3}{5n+4}, \lim_{n\to\infty} \frac{n^2+n}{n^3-n} \right) = \left(\frac{2}{5}, 0 \right)$

b) $\lim_{n\to\infty} x_n = \left(\lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n}, \lim_{n\to\infty} \frac{1}{n} - 1, \lim_{n\to\infty} \frac{\cancel{7^n}\left(\left(\frac{3}{7}\right)^n+1\right)}{\cancel{7^n}\left(\left(-\frac{3}{7}\right)^n+1+\frac{1}{7^n}\right)} \right)$

$= \left(\frac{1}{e}, -1, 1 \right)$

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(18) $\lim_{(x,y)\to(\infty,a)} \left(1+\frac{1}{x}\right)^{\frac{x^2}{x+y}} \overset{1^\infty}{=} \lim_{\substack{x\to\infty \\ y\to a}} e^{\ln\left(1+\frac{1}{x}\right)^{\frac{x^2}{x+y}}} = e^{l}$

$l = \lim_{\substack{x\to\infty \\ y\to a}} \ln\left(1+\frac{1}{x}\right)^{\frac{x^2}{x+y}} = \lim_{\substack{x\to\infty \\ y\to a}} \left(\underbrace{\left(\frac{x}{x+y}\right)}_{l_1} \cdot \underbrace{\ln\left(1+\frac{1}{x}\right)^x}_{l_2} \right)$

$l_1 = \lim_{\substack{x\to\infty \\ y\to a}} \ln\left(1+\frac{1}{x}\right)^{\frac{1}{x}} = \ln e = 1$

$l_2 = \lim_{\substack{x\to\infty \\ y\to a}} \left(\frac{x}{x+y}\right) = \lim_{\substack{x\to\infty \\ y\to a}} \frac{1}{1+\frac{y}{x}} = 1$

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(19) Ukoliko postoji izračunati

$$l = \lim_{(x,y)\to(0,0)} (x+y)\cdot \sin\frac{1}{x}\cdot\sin\frac{1}{y}$$

→ pokušamo teoremom o ukliještenju da dokažemo

→ kada hoćemo nešto oko 0 → apsol. vr.

$$0 \le \left|(x+y)\cdot\sin\frac{1}{x}\cdot\sin\frac{1}{y}\right| = |x+y|\cdot\left|\sin\frac{1}{x}\right|\cdot\left|\sin\frac{1}{y}\right|$$

$$|x+y|\left|\sin\frac{1}{x}\right|\left|\sin\frac{1}{y}\right| \le \underset{\swarrow\ 0}{|x+y|}$$

$$|x| \le a \rightarrow -a \le x \le a$$

$\Rightarrow$ iz ovoga $\Rightarrow l=0$ (teorema o ukliještenju)

(20) Da li postoji $\lim_{\substack{x\to+\infty\\ y\to+\infty}} \frac{2x+3y}{x^2+xy+y^2} = l$

Ideja: $0 \le \left|\frac{2x+3y}{x^2+xy+y^2}\right| \le \frac{2x+3y}{xy} = \underbrace{\frac{2}{y}+\frac{3}{x}}_{\swarrow\ 0}$

$\Rightarrow l \to 0$ (teorema o ukliještenju)

(21) $\lim_{\substack{x\to 0\\ y\to 0}} \frac{x}{y}$ $\quad a_n = \left(0, \frac{1}{n}\right)$ → za b_n ne možemo $\left(\frac{1}{n}, 0\right)$ jer $\frac{x}{y}$ $y \ne 0$

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} \frac{0}{\frac{1}{n}} = 0$$

$$b_n = \left(\frac{1}{n}, \frac{1}{n}\right)$$

$$\lim_{n\to\infty} f(b_n) = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n}} = 1$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = (0,0)$$

$\lim_{n\to\infty} f(a_n) \ne \lim_{n\to\infty} f(b) \Rightarrow$ granična vrijednost ne postoji

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(22) $\lim\limits_{\substack{x\to+\infty \\ y\to+\infty}} x^2 \cdot e^{-(x-y^2)}$

Uočimo 2 niza: $x_n=(n^2,n)$ i $y_n=(5n^2,n)$

$\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = (\infty,\infty)$

$\lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} n^4 \cdot e^{-(n^2-n^2)} = \lim\limits_{n\to\infty} n^4 = \infty$

$\lim\limits_{n\to\infty} f(y_n) = \lim\limits_{n\to\infty} 25\cdot n^4 \cdot e^{-(5n^2-n^2)} = \lim\limits_{n\to\infty} \frac{25n^4}{e^{4n^2}} = 0$

$\Rightarrow$ Ne postoji $\lim\limits_{\substack{x\to\infty \\ y\to\infty}} f(x,y)$

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(23) Naći $\lim\limits_{\substack{x\to 0 \\ y\to 0}} \left(x + y\cdot \sin\frac{1}{x}\right)$

$0 \leq \left|x+y\cdot\sin\frac{1}{x}\right| \leq |x| + \left|y\cdot\sin\frac{1}{x}\right| = |x|+|y|\cdot\left|\sin\frac{1}{x}\right|$

$\leq |x| + |y| \quad (1)$

$\lim\limits_{(x,y)\to(0,0)} (|x|+|y|) = 0 \quad (2)$

Iz (1) i (2) slijedi da je $\lim\limits_{\substack{x\to 0 \\ y\to 0}} \left|x+y\sin\frac{1}{x}\right| = 0$

pa je i $\lim\limits_{\substack{x\to 0 \\ y\to 0}} \left(x+y\sin\frac{1}{x}\right) = 0$

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(24) Dokazati da $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ nije neprekidna u tački (0,0)

I način: $\rightarrow$ pokažemo da ne postoji $\lim\limits_{\substack{x\to 0 \\ y\to 0}} f(x,y)$ ~~pa~~ (kao u zadatku 15), pa na osnovu toga slijedi da fja nije neprekidna u tački (0,0)

II način → na osnovu istog zadatka samo pod b

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right)=1$$

$$\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right)=-1$$

---

25. Ispitati neprekidnost f-je $f(x,y)=\begin{cases}\frac{x^2\cdot y}{x^4+y^2} & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

Za $(x,y)\neq(0,0)$ f-ja je neprekidna kao količnik dvije neprekidne funkcije u oblasti definisanosti.

Pokažimo da f nije neprekidna u $(0,0)$

Neka je $x_n=\left(\frac{1}{n},\frac{1}{n}\right)$, a $y_n=\left(\frac{1}{n},\frac{1}{n^2}\right)$.

Tada je:

1° $\lim\limits_{n\to\infty} f(x_n)=\lim\limits_{n\to\infty} y_n=(0,0)$

2° $\lim\limits_{n\to\infty} f(x_n)=\lim\limits_{n\to\infty} f\left(\frac{1}{n},\frac{1}{n}\right)=\lim\limits_{n\to\infty}\frac{\frac{1}{n^2}\cdot\frac{1}{n}}{\frac{1}{n^4}+\frac{1}{n^2}}=$

$=\lim\limits_{n\to\infty}\frac{n}{1+n^2}=0$

$\lim\limits_{n\to\infty} f(y_n)=\lim\limits_{n\to\infty} f\left(\frac{1}{n},\frac{1}{n^2}\right)=\lim\limits_{n\to\infty}\frac{\frac{1}{n^2\cdot n^2}}{\frac{1}{n^4}+\frac{1}{n^4}}=\frac{1}{2}$

→ $\lim\limits_{n\to\infty} f(x_n)\neq\lim\limits_{n\to\infty} f(y_n)$

→ iz 1° i 2° zaključujemo da granična vrijednost funkcije ne postoji u $(0,0)$; da postoji, morali bi biti jednaki $\lim\limits_{n\to\infty} f(x_n)$ i $\lim\limits_{n\to\infty} f(y_n)$

⇒ pa f-ja $f(x,y)$ nije neprekidna u $(0,0)$

AGATHA RUIZ DE LA PRADA

26. Ispitati neprekidnost funkcije:

$$f(x,y)=\begin{cases}\dfrac{x^2-y^2}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

* u tačkama $(x,y)\neq(0,0)$ f-ja f je neprekidna kao količnik neprekidnih funkcija

$$\left|\cancel{\frac{x^2-y^2}{\sqrt{x^2+y^2}}}\right| \quad \left|\frac{x^2-y^2}{\sqrt{x^2+y^2}}\right| \leq \left|\frac{x^2+y^2}{\sqrt{x^2+y^2}}\right|$$

apsol.

$$-\frac{x^2+y^2}{\sqrt{x^2+y^2}} \leq \frac{x^2-y^2}{\sqrt{x^2+y^2}} \leq \frac{x^2+y^2}{\sqrt{x^2+y^2}} \xrightarrow[\substack{x\to0\\y\to0}]{} 0$$

$0 \leftarrow$ ($x\to0$, $y\to0$)

$$L: \frac{x^2-y^2}{\sqrt{x^2+2xy+y^2}} = \frac{x^2-y^2}{x+y} = (x-y)$$

$$\lim_{\substack{x\to0\\y\to0}} \frac{x^2-y^2}{\sqrt{x^2+y^2}} = 0, \text{ tj } \lim_{\substack{x\to0\\y\to0}} f(x,y) = f(0,0) = 0$$

$\Rightarrow$ f-ja jeste neprekidna u $(0,0)$

$$\sqrt{x^2+y^2} \leq \sqrt{x^2+2xy+y^2} \rightarrow \frac{1}{\sqrt{x^2+y^2}} \geq \frac{1}{\sqrt{x^2+2xy+y^2}}$$

$\cancel{x^2+y^2}$

$$\frac{x^2-y^2}{\sqrt{x^2+y^2}} \geq \frac{x^2-y^2}{\sqrt{x^2+2xy+y^2}} = \frac{(x-y)\cancel{(x+y)}}{\cancel{x+y}} = x-y$$

$$\frac{x^2-y^2}{\sqrt{x^2+y^2}} \leq \frac{x^2-y^2}{\sqrt{x^2-2xy+y^2}} = \frac{\cancel{(x-y)}(x+y)}{\cancel{x-y}} = x+y$$

---

27. Ispitati neprekidnost funkcije $f(x,y)$:

$$f(x,y)=\begin{cases}\dfrac{\sin x\cdot\sin y}{x\cdot y}, & x\cdot y\neq0\\ 1, & x\cdot y=0\end{cases}$$ u tačkama

$O(0,0)$ i $A(1,0)$; posebno po svakoj promjenljivoj kao i po promjenljivoj (x,y).

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{\sin x}{x} \cdot \frac{\sin y}{y} = 1 \cdot 1 = 1 = f(0,0)$$

$\Rightarrow$ funkcija $f(x,y)$ je neprekidna u tački $O(0,0)$ po promjenljivoj (x,y) $\Rightarrow$ da je fja f neprekidna u tački $O(0,0)$ i po promjenljivoj x i po promjenljivoj y

$$\lim_{(x,y)\to(1,0)} f(x,y) = \lim_{(x,y)\to(1,0)} \frac{\sin x \cdot \sin y}{x \cdot y} = \lim_{(x,y)\to(1,0)} \frac{\sin x}{x} \cdot \lim_{\substack{x\to 1\\ y\to 0}} \frac{\sin y}{y}$$

$$= \frac{\sin 1}{1} \cdot 1 = \sin 1 \neq f(1,0) \Rightarrow$$

$\Rightarrow$ fja f nije neprekidna u tački $A(1,0)$ po promjenljivoj (x,y)

Po pojedinačnim promjenljivim: $y=0$:

$$\lim_{x\to 1} f(x,0) = 1 \quad (x \cdot y = 0) \quad = f(1,0) \Rightarrow$$

$\Rightarrow$ fja f je neprekidna u tački $A(1,0)$ po promjenljivoj x

$x=1$

$$\lim_{y\to 0} f(1,y) = \begin{cases} \dfrac{\sin 1 \cdot \sin y}{1 \cdot y}, & y \neq 0 \\ 1, & y = 0 \end{cases}$$

$$\lim_{y\to 0} f(1,y) = \lim_{y\to 0} \sin 1 \cdot \frac{\sin y}{y} = \sin 1 \cdot 1 \neq 1 = f(1,0)$$

$\Rightarrow$ fja f nije neprekidna u tački $A(1,0)$ po promjenljivoj y.

28. Dokazati neprekidnost funkcije

$$f(x,y) = \begin{cases} \frac{x\cdot y}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$ u tački

$O(0,0)$ po svakoj promjenljivoj posebno i njenu neprekidnost u istoj tački kao fju 2 promjenljive.

$y=0$: $f(x,0) = \begin{cases} \frac{x\cdot 0}{x^2+0^2}, & x\neq 0 \to (x,0)\neq(0,0) \\ 0, & x=0 \quad (x,0)=(0,0) \end{cases}$

$g(x) = f(x,0) = 0, \ \forall x\in\mathbb{R}$

$\lim_{x\to 0} f(x,0) = 0 = f(0,0) \Rightarrow f$ je neprekidna u tački $O(0,0)$ po promjenljivoj ~~y~~

$x=0$: $f(0,y) = \begin{cases} \frac{0\cdot y}{0^2+y^2}, & y\neq 0 \\ 0, & y=0 \to (y,0)=(0,0) \end{cases}$

$f(0,y)=0, \ \forall y\in\mathbb{R}$

$\lim_{y\to 0} f(0,y) = 0 = f(0,0) \Rightarrow$ fja f je neprekidna u tački $O(0,0)$ po promjenljivoj y

Neka je $x_n = \left(\frac{1}{n},\frac{1}{n}\right)$ i $y_n = \left(\frac{1}{n},-\frac{1}{n}\right)$.

Tada je $\lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(\frac{1}{n},\frac{1}{n}\right) = (0,0)$

1° $\lim_{n\to\infty} y_n = \lim_{n\to\infty}\left(\frac{1}{n},-\frac{1}{n}\right) = (0,0)$

2° $\begin{cases} \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f\left(\frac{1}{n},\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n^2}+\frac{1}{n^2}} = \frac{1}{2} \\ \lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} f\left(\frac{1}{n},-\frac{1}{n}\right) = \lim_{n\to\infty} \frac{-\frac{1}{n^2}}{\frac{1}{n^2}+\frac{1}{n^2}} = -\frac{1}{2} \end{cases}$

Iz 1° i 2° $\Rightarrow$ ne postoji $\lim_{(x,y)\to(0,0)} f(x,y)$ tada

29. Ispitati neprekidnost funkcije:

$$f(x,y)=\begin{cases}\dfrac{\sin(x+y)}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

→ u tačkama Euklidove ravni za koje je $(x,y)\neq(0,0)$ funkcija f je neprekidna kao kompozicija neprekidnih funkcija. Pokažimo da fja f ima prekid u $(0,0)$.

$x_n=\left(0,\frac{1}{n}\right)$, $y_n=\left(0,-\frac{1}{n}\right)$

može i $\left(\frac{1}{n},\frac{1}{n}\right)$ $\left(-\frac{1}{n},\frac{1}{n}\right)$

$\lim\limits_{n\to\infty} x_n=\lim\limits_{n\to\infty} y_n=(0,0)$

$$\lim_{n\to\infty} f(x_n)=\lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}}=\left[\begin{matrix}\frac{1}{n}=m\\ n\to\infty\\ m\to 0\end{matrix}\right]=\lim_{m\to 0}\frac{\sin m}{m}=1$$

$$\lim_{n\to\infty} f(y_n)=\lim_{n\to\infty}\frac{\sin\left(-\frac{1}{n}\right)}{\frac{1}{n}}=-\lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}}=-1$$

$\lim\limits_{n\to\infty} f(x_n)\neq\lim\limits_{n\to\infty} f(y_n)\Rightarrow$ ne postoji $\lim\limits_{(x,y)\to(0,0)} f(x,y)$

pa fja f u $O(0,0)$ ima prekid

---

30. Ispitati neprekidnost funkcije:

$$f(x,y)=\begin{cases}\dfrac{\sin^2(x+y)}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

→ Funkcija je neprekidna u tačkama za koje je $(x,y)\neq(0,0)$ kao kompozicija neprekidnih funkcija.

$$0 \le \sin^2(x+y) \le |x+y|^2 = (x+y)^2 = x^2+2xy+y^2 \le$$

$$\le x^2 + \underline{x^2+y^2} + y^2$$

$$0 \le \sin^2(x+y) \le 2(x^2+y^2)$$

$$0 \le \frac{\sin^2(x+y)}{\sqrt{x^2+y^2}} \le 2\sqrt{x^2+y^2} \quad (2)$$

$$\lim_{(x,y)\to(0,0)} 2\sqrt{x^2+y^2} = 0 \quad (3)$$

→ iz (2) i (3) slijedi da je $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$

⇒ pa je fja f neprekidna u tački $(0,0)$

$$\boxed{(x-y)^2 \ge 0 \longrightarrow x^2-2xy+y^2 \ge 0 \qquad 2xy \le x^2+y^2}$$

Zapamtiti! često se koristi

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31. Ispitati neprekidnost fje: → Težina na kolokvijumu

$$f(x,y) = \begin{cases} \dfrac{x^2 \sin y}{x^4+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

2 stvari:

1) $\sin y$ liči na y kad težimo 0

2) $\dfrac{xy}{x^2+y^2}$ liči na $\dfrac{x^2 \sin y}{x^4+y^2}$

→ Pokazaćemo da gr. vr. $\lim_{(x,y)\to(0,0)} \dfrac{x^2 \sin y}{x^4+y^2}$ ne postoji

$$\lim_{(x,y)\to(0,0)} \frac{x^2 \sin y}{x^4+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2 \cdot y}{x^4+y^4} \cdot \frac{\sin y}{y} =$$

2 gr. vr. → 1 postoji, druga ne postoji ⇒ u proizvodu gr. vr. ne postoji

$$= \lim_{(x,y)\to(0,0)} \frac{x^2 \cdot y}{x^4+y^2} \quad \lim_{(x,y)\to(0,0)} \frac{\sin y}{y} \to 1$$

→ dovoljno pokazati da ne postoji gr. vr. $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2}$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2} = \lim_{(x,y)\to(0,0)} g(x,y)$$

↳ dokazujemo da ne postoji tako što nađemo 2 niza da teže 0, ali kad uvrstimo ne teže istoj vr, odnosno

$a_n = (0, \frac{1}{n})$

$b_n = (\frac{1}{n}, \frac{1}{n^2})$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = (0,0) \quad (1)$$

$$\lim_{n\to\infty} g(0, \tfrac{1}{n}) = \lim_{n\to\infty} \frac{0 \cdot \frac{1}{n}}{0 + \frac{1}{n^2}} = 0 \quad (2)$$

$$\lim_{n\to\infty} g(b_n) = \lim_{n\to\infty} \frac{\frac{1}{n^2} \cdot \frac{1}{n^2}}{\frac{1}{n^4} + \frac{1}{n^4}} = \frac{1}{2} \quad (3)$$

→ iz (1), (2) i (3) na osnovu Hajneove definicije granične vrijednosti ne postoji $\lim_{(x,y)\to(0,0)} g(x)$.

$c_n = (\frac{a}{n}, \frac{b}{n}) \to (0,0)$

$$\lim_{n\to\infty} g(c_n) = \lim_{n\to\infty} \frac{\frac{a^2}{n^2} \cdot \frac{b}{n}}{\frac{a^4}{n^4} + \frac{b^2}{n^2}} = \lim_{n\to\infty} \frac{n a^2 b}{a^4 + b^2 n^2} = 0$$